ENGR 2420: Introduction to Microelectronic Circuits

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5 Differential Pair

In this section, we shall consider the circuit shown in Fig. 1, which is called the *differential pair.*[1] This venerable circuit configuration has a long history—it can be traced back to Fig. 3 of British Patent Specification 482,740 filed by A. D. Blumlein on July 4, 1936, where it appears implemented using vacuum tubes. Since its inception in the vacuum-tube era, this configuration has been used as the input stage of most of the operational amplifiers and comparators that have been designed. As we shall see, it is very sensitive to changes in the *difference between* its input voltages, V_1 and V_2, while being insensitive to their absolute levels.

Consider the circuit shown in Fig. 1. At any level of bias current, I_b, Kirchhoff's Current Law (KCL) applied at the common-source node, V, implies that, in the steady state,

$$I_\mathrm{b} = I_1 + I_2. \tag{1}$$

That turns out to be the secret of the differential pair's success. We are able to keep the incremental properties (e.g., the incremental transconductance gains) of M_1 and M_2 in this circuit nearly constant, despite large changes in the input voltages, by keeping the sum of the two output currents, $I_1 + I_2$, fixed at a constant value given by the bias current, I_b. We arrange this constraint by allowing the voltage on the common-source node, V, to move up and down with the input voltages, V_1 and V_2, in such a way that $I_1 + I_2 = I_\mathrm{b}$. Suppose that we increase V_1 relative to V_2; then, I_1 would increase, causing the sum $I_1 + I_2$ to increase transiently. The amount by which this sum exceeds I_b charges up node V, reducing both I_1 and I_2 until the sum is just equal to I_b. Conversely, if we were to decrease V_1 relative to V_2, then I_1 would decrease, causing a deficit in the sum $I_1 + I_2$. The amount by which I_b exceeds this sum would discharge node V, increasing both I_1 and I_2 until the sum is again equal to I_b. Similar conditions hold for increases and decreases in V_2.

5.1 Weak-Inversion Operation

We shall now analyze the nMOSdifferential pair characteristics under the assumption of weak-inversion operation. As we have seen, each of the differential pair's output currents is bounded above by the bias current. Consequently, if M_1 and M_2 remain in saturation and the bias current is a weak-inversion current, then M_1 and M_2 must also operate in weak inversion. Consequently, we have that the output currents are given by

$$I_1 = SI_\mathrm{s}e^{(\kappa(V_1 - V_\mathrm{T0}) - V)/U_\mathrm{T}} \tag{2}$$

and

$$I_2 = SI_\mathrm{s}e^{(\kappa(V_2 - V_\mathrm{T0}) - V)/U_\mathrm{T}}. \tag{3}$$

[1]This circuit is sometimes also referred to as the *source-coupled pair* or as the *long-tailed pair*.

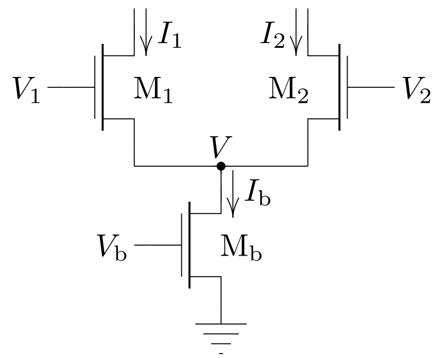

Figure 1: An nMOSdifferential pair circuit.

Now, we can substitute Eq. 2 and Eq. 3 into Eq. 1 to find that

$$\begin{aligned} I_{\rm b} &= SI_{\rm s}e^{(\kappa(V_1-V_{\rm T0})-V)/U_{\rm T}} + SI_{\rm s}e^{(\kappa(V_2-V_{\rm T0})-V)/U_{\rm T}} \\ &= SI_{\rm s}e^{-(\kappa V_{\rm T0}+V)/U_{\rm T}}\left(e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}\right), \end{aligned}$$

which upon rearrangement becomes

$$SI_{\rm s}e^{-(\kappa V_{\rm T0}+V)/U_{\rm T}} = \frac{I_{\rm b}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}}. \tag{4}$$

By substituting Eq. 4 into Eq. 2 and into Eq. 3, we have that, in weak inversion, the output currents are given by

$$I_1 = I_{\rm b}\cdot\frac{e^{\kappa V_1/U_{\rm T}}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}} = \frac{I_{\rm b}}{1+e^{-\kappa(V_1-V_2)/U_{\rm T}}} \tag{5}$$

and

$$I_2 = I_{\rm b}\cdot\frac{e^{\kappa V_2/U_{\rm T}}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}} = \frac{I_{\rm b}}{1+e^{\kappa(V_1-V_2)/U_{\rm T}}}, \tag{6}$$

respectively. Using these equations, we can obtain an expression for the differential output current, $I_1 - I_2$, which is often that which we care most about. By subtracting Eq. 6 from Eq. 5, we have that

$$\begin{aligned} I_1 - I_2 &= I_{\rm b}\cdot\frac{e^{\kappa V_1/U_{\rm T}}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}} - I_{\rm b}\cdot\frac{e^{\kappa V_2/U_{\rm T}}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}} \\ &= I_{\rm b}\cdot\frac{e^{\kappa V_1/U_{\rm T}} - e^{\kappa V_2/U_{\rm T}}}{e^{\kappa V_1/U_{\rm T}} + e^{\kappa V_2/U_{\rm T}}}. \end{aligned} \tag{7}$$

At this point, we shall find it useful to express the differential output current in terms of the *common-mode* input voltage, $V_{\rm cm}$, and the *differential-mode* input voltage, $V_{\rm dm}$, which are given by

$$V_{\rm cm} = \tfrac{1}{2}(V_1 + V_2) \quad \text{and} \quad V_{\rm dm} = V_1 - V_2, \tag{8}$$

respectively. By solving these equations simultaneously for the original input voltages, we can express V_1 and V_2 in terms of $V_{\rm cm}$ and $V_{\rm dm}$ as

$$V_1 = V_{\rm cm} + \tfrac{1}{2}V_{\rm dm} \quad \text{and} \quad V_2 = V_{\rm cm} - \tfrac{1}{2}V_{\rm cm}, \tag{9}$$

respectively. By substituting Eq. 9 into Eq. 7, we have that

$$\begin{aligned} I_1 - I_2 &= I_\text{b} \cdot \frac{e^{\kappa V_\text{cm}/U_\text{T}} e^{\kappa V_\text{dm}/2U_\text{T}} - e^{\kappa V_\text{cm}/U_\text{T}} e^{-\kappa V_\text{dm}/2U_\text{T}}}{e^{\kappa V_\text{cm}/U_\text{T}} e^{\kappa V_\text{dm}/2U_\text{T}} + e^{\kappa V_\text{cm}/U_\text{T}} e^{-\kappa V_\text{dm}/2U_\text{T}}} \\ &= I_\text{b} \cdot \frac{e^{\kappa V_\text{cm}/U_\text{T}}}{e^{\kappa V_\text{cm}/U_\text{T}}} \cdot \frac{e^{\kappa V_\text{dm}/2U_\text{T}} - e^{-\kappa V_\text{dm}/2U_\text{T}}}{e^{\kappa V_\text{dm}/2U_\text{T}} + e^{-\kappa V_\text{dm}/2U_\text{T}}} \\ &= I_\text{b} \cdot \frac{e^{\kappa V_\text{dm}/2U_\text{T}} - e^{-\kappa V_\text{dm}/2U_\text{T}}}{e^{\kappa V_\text{dm}/2U_\text{T}} + e^{-\kappa V_\text{dm}/2U_\text{T}}} \\ &= I_\text{b} \tanh \frac{\kappa V_\text{dm}}{2U_\text{T}}. \end{aligned} \tag{10}$$

Finally, we shall develop an explicit expression for the common-source node voltage, V. If M_b operates in weak inversion and is saturated, we have that

$$I_\text{b} = SI_\text{s} e^{\kappa(V_\text{b} - V_\text{T0})/U_\text{T}}.$$

By dividing this equation by Eq. 4, we find that

$$e^{\kappa V_\text{b}/U_\text{T}} e^{V/U_\text{T}} = e^{\kappa V_1/U_\text{T}} + e^{\kappa V_2/U_\text{T}},$$

which we can solve for the common-source node voltage to find that

$$V = U_\text{T} \log\left(e^{\kappa V_1/U_\text{T}} + e^{\kappa V_2/U_\text{T}}\right) - \kappa V_\text{b}.$$

If V_1 exceeds V_2 by only a few U_T, then the first term in parentheses quickly renders the second negligible in comparison, and we have that

$$V \approx U_\text{T} \log e^{\kappa V_1/U_\text{T}} - \kappa V_\text{b} = \kappa\left(V_1 - V_\text{b}\right).$$

On the other hand, if V_2 exceeds V_1 by only a few U_T, the second term in parentheses renders the first negligible in comparison, and we have that

$$V \approx U_\text{T} \log e^{\kappa V_2/U_\text{T}} - \kappa V_\text{b} = \kappa\left(V_2 - V_\text{b}\right).$$

By combining these results, we have that the common-source node voltage is given by

$$\begin{aligned} V &= U_\text{T} \log\left(e^{\kappa V_1/U_\text{T}} + e^{\kappa V_2/U_\text{T}}\right) - \kappa V_\text{b} \\ &\approx \kappa\left(\max(V_1, V_2) - V_\text{b}\right). \end{aligned} \tag{11}$$

It is relatively easy to see that the approximation is a lower-bound on V and that the maximum error occurs when $V_1 = V_2$ and is equal to $U_\text{T} \log 2$, which is quite small.

5.2 Strong-Inversion Operation

Now, we shall analyze the behavior of the differential pair assuming that all three transistors are saturated and operate in strong inversion. Unfortunately, the bias transistor's operating in strong inversion does not guarantee that M_1 and M_2 also operate in strong inversion—as the bias current is steered to one side, it steals current from the other, eventually resulting in the loosing side entering moderate and eventually weak inversion. Consequently, we shall need to examine the domain of validity our results.

If M_1 and M_2 both operate in strong inversion and in saturation, then we have that I_1 is given by

$$I_1 = \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\left(\kappa\left(V_1 - V_\mathrm{T0}\right) - V\right)^2, \tag{12}$$

and that I_2 is given by

$$I_2 = \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\left(\kappa\left(V_2 - V_\mathrm{T0}\right) - V\right)^2. \tag{13}$$

Similarly, if M_b also operates in strong inversion and in saturation, then we have that I_b is given by

$$I_\mathrm{b} = \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\left(\kappa\left(V_\mathrm{b} - V_\mathrm{T0}\right)\right)^2. \tag{14}$$

By substituting Eq. 12, Eq. 13, and Eq. 14 into Eq. 1 and dividing both sides by the common factor of $SI_\mathrm{s}/4U_\mathrm{T}^2$, we find that

$$\kappa^2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 = \left(\kappa\left(V_1 - V_\mathrm{T0}\right) - V\right)^2 + \left(\kappa\left(V_2 - V_\mathrm{T0}\right) - V\right)^2.$$

By expanding this equation and collecting powers of V, we obtain a quadratic equation for V, which is given by

$$0 = \underbrace{2}_{a}V^2 - \underbrace{2\kappa\left(V_1 + V_2 - 2V_\mathrm{T0}\right)}_{b}V + \underbrace{\kappa^2\left(\left(V_1 - V_\mathrm{T0}\right)^2 + \left(V_2 - V_\mathrm{T0}\right)^2 - \left(V_\mathrm{b} - V_\mathrm{T0}\right)^2\right)}_{c}. \tag{15}$$

Before we apply the quadratic formula indiscriminately to solve Eq. 15, we shall work to simplify its discriminant; doing so, we obtain

$$\begin{aligned} b^2 - 4ac &= 4\kappa^2\left(V_1 + V_2 - 2V_\mathrm{T0}\right)^2 - 8\kappa^2\Big(\left(V_1 - V_\mathrm{T0}\right)^2 + \left(V_2 - V_\mathrm{T0}\right)^2 - \left(V_\mathrm{b} - V_\mathrm{T0}\right)^2\Big) \\ &= 4\kappa^2\left(2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 - \left(V_1 - V_2\right)^2\right). \end{aligned}$$

By applying the quadratic formula to Eq. 15 and making use of this result, we have that the possible solutions for V are given by

$$V = \frac{\kappa}{2}\left(V_1 + V_2 - 2V_\mathrm{T0} \pm \sqrt{2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 - \left(V_1 - V_2\right)^2}\right). \tag{16}$$

Which one of these roots should we pick? In order for M_1 to operate in strong inversion, we must have that

$$\kappa\left(V_1 - V_\mathrm{T0}\right) - V > 0 \quad\Longrightarrow\quad V < \kappa\left(V_1 - V_\mathrm{T0}\right).$$

Similarly, in order for M_2 to operate in strong inversion, we must have that

$$\kappa\left(V_2 - V_\mathrm{T0}\right) - V > 0 \quad\Longrightarrow\quad V < \kappa\left(V_2 - V_\mathrm{T0}\right).$$

By adding these two inequalities and dividing both sides by 2, we find that in order for both M_1 and M_2 to operate in strong inversion, we require of V that

$$V < \frac{\kappa}{2}\left(V_1 + V_2 - 2V_\mathrm{T0}\right),$$

which implies that we must choose the root with the negative sign in Eq. 16.

By substituting Eq. 15 into Eq. 12, we obtain an expression for I_1, which is given by

$$\begin{aligned}
I_1 &= \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\left(\kappa\left(V_1 - V_\mathrm{T0}\right) - \frac{\kappa}{2}\left(V_1 + V_2 - 2V_\mathrm{T0} - \sqrt{2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 - \left(V_1 - V_2\right)^2}\right)\right)^2 \\
&= \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\cdot\frac{\kappa^2}{4}\left(\left(V_1 - V_2\right) + \sqrt{2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 - \left(V_1 - V_2\right)^2}\right)^2 \\
&= \frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\cdot\frac{\kappa^2}{4}\left(2\left(V_1 - V_2\right)\sqrt{2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2 - \left(V_1 - V_2\right)^2} + 2\left(V_\mathrm{b} - V_\mathrm{T0}\right)^2\right) \\
&= \frac{1}{2}\cdot\frac{SI_\mathrm{s}}{4U_\mathrm{T}^2}\left(\kappa\left(V_\mathrm{b} - V_\mathrm{T0}\right)\right)^2\left(1 + \frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\sqrt{2 - \left(\frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\right)^2}\right) \\
&= \frac{I_\mathrm{b}}{2}\left(1 + \frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\sqrt{2 - \left(\frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\right)^2}\right). \qquad (17)
\end{aligned}$$

Similarly, by substituting Eq. 15 into Eq. 13, we obtain an expression for I_2, which is given by

$$I_2 = \frac{I_\mathrm{b}}{2}\left(1 - \frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\sqrt{2 - \left(\frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\right)^2}\right). \qquad (18)$$

Finally, by subtracting Eq. 18 from Eq. 17, we can find an expression for the differential output current, $I_1 - I_2$, given by

$$I_1 - I_2 = I_\mathrm{b}\cdot\frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\sqrt{2 - \left(\frac{V_1 - V_2}{V_\mathrm{b} - V_\mathrm{T0}}\right)^2}. \qquad (19)$$

What does this formidable expression mean (other than large-signal analysis for strong-inversion CMOS circuits is not for the faint of heart)? We'll leave that for you to ponder... Good luck!