ENGR 2420: Introduction to Microelectronic Circuits

February 7, 2020

2 Incremental Driving-Point Transfer Functions

In this section, we shall examine two important incremental relationships between the amount of current flowing into some port of a nonlinear circuit and the voltage across that port, as shown in Fig. 1a. The first of these relationships is called the incremental *driving-point conductance* of a port, which is a measure of how much additional current flows into the port in response to a small change in the voltage across it. The second of these relationships, called the incremental *driving-point resistance*, is closely related to the driving-point conductance—it tells by how much the voltage across a port will change in response to a small change in the current flowing into the port. The incremental driving-point resistance of a port is what we commonly refer to as *the (incremental) resistance seen looking into the port*. Note that, for a driving-point conductance, the excitation is a small change in the voltage across the port and the response is the resulting small change in the current flowing into the port. On the other hand, for a driving-point resistance, the excitation is a small change in the current flowing into the port and the response is the resulting small change in the voltage across the port.

Formally, the driving-point conductance, G_{dp}, of a port is defined as the partial derivative of the current flowing into the port with respect to the voltage across the port,

$$G_{\text{dp}} \equiv \frac{\partial I}{\partial V}.$$

Likewise, the driving-point resistance, R_{dp}, of a port is formally defined as the partial derivative of the voltage across the port with respect to the current flowing into the port,

$$R_{\text{dp}} \equiv \frac{\partial V}{\partial I}.$$

Note that each of these quantities is the reciprocal of the other, so we can readily determine one if we know the other and we are, thus, free to excite the port with either a voltage source or a current source to determine both driving-point transfer functions, whichever is more convenient.

Figure 1b illustrates the process of determining the driving-point resistance or driving-point conductance of a port by injecting a small current, δI into the port and measuring the resulting change in the voltage across the port, δV. The incremental driving-point resistance of the port is given by $\delta V/\delta I$ and its incremental driving-point conductance is given by $\delta I/\delta V$. Similarly, Fig. 1c illustrates the process of determining the driving-point resistance or driving-point conductance of a port by forcing a small change in the voltage across the port and measuring the resulting change in the current flowing into the port. In this case, we first measure the voltage across the port and adjust a test voltage source to have precisely this value. We then connect this test voltage source across the port. No current will flow between the voltage source and the circuit at this point, because we adjusted the

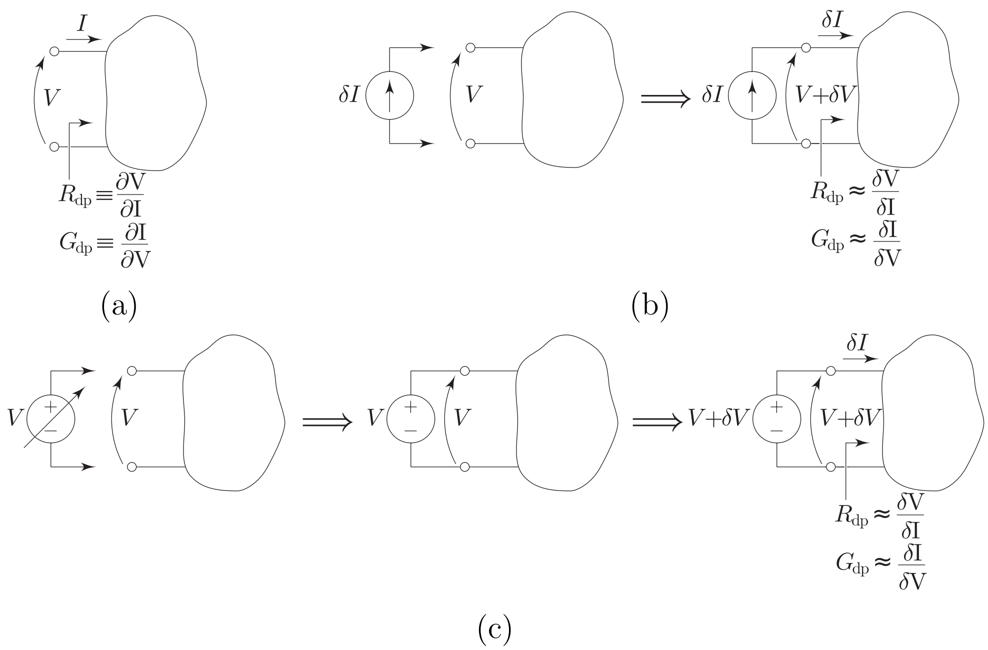

Figure 1: Incremental *driving-point resistance* and *driving-point conductance* of a port. (a) We can determine each of these incremental transfer functions by injecting a small test current, δI, into the port and measuring the resulting voltage change, δV. (b) We can also determine them by adjusting the value of a test voltage source to match exactly the port voltage, V, attaching the voltage source to the port, and changing its value by a small amount, δV, and measuring the current, δI, that goes into the port as a result.

voltage sources value to that of the open-circuited port voltage. If the test voltage source had a larger value than the open-circuited port voltage, the circuit would draw a current from the voltage source. If the test source's value were smaller than the open-circuited port voltage, the circuit would supply a current to the test source. We then change the test voltage source by a small amount, δV, and measure the resulting current drawn by the circuit, δI. In this case, the incremental driving-point conductance of the port is given by $\delta I/\delta V$ and its incremental driving-point resistance is given by $\delta V/\delta I$.

An important special case of an incremental port driving-point resistance or port driving-point conductance occurs when one side of the port is connected to ground. In this case, the voltage involved in the driving-point resistance or driving-point conductance is the node voltage, as illustrated in Fig. 2a. These quantities are what we commonly refer to, respectively, as the (incremental) *resistance of a node* and the (incremental) *conductance of a node*. Figure 2b illustrates how we can determine these quantities by injecting a small test current, δI, onto a node and measuring the resulting node-voltage change, δV. Figure 2c illustrates how we can determine these quantities by forcing the node voltage from its quiescent value by a small amount, δV, and measuring how much current, δI flows into the node as a result. In either case, the ratio of δV to δI gives the incremental impedance of the node and the ratio of δI to δV gives the incremental admittance of the node.

In a similar manner, we can define the incremental driving-point resistance and incre-

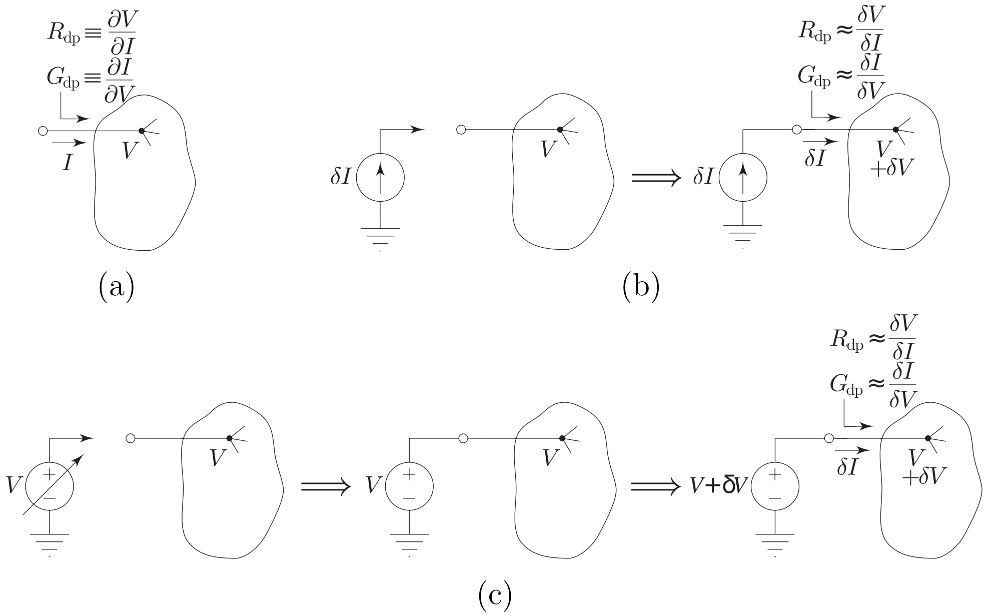

Figure 2: Incremental driving-point resistance and conductance of a node. (a) We can determine each of these incremental transfer functions by injecting a small test current, δI, into the node and measuring the resulting node-voltage change, δV. (b) We can also determine them by adjusting the value of a test voltage source to match exactly the node voltage, V, attaching the voltage source to the port, and changing its value by a small amount, δV, and measuring the current, δI, that goes into the node as a result.

mental driving-point conductance looking into a terminal of a nonlinear device, X, as shown in Fig. 3, given by

$$R_{\mathrm{X}} \equiv \frac{\partial V}{\partial I} \quad \text{and} \quad G_{\mathrm{X}} \equiv \frac{\partial I}{\partial V},$$

respectively. For a sufficiently small change in the terminal voltage, V, the change in the current, I, flowing into the terminal will be given approximately by

$$\delta I \approx G_{\mathrm{X}} \delta V.$$

Here, *sufficiently small* basically means small enough that the second and higher order terms in the Taylor-series expansion of X's current–voltage characteristics about the point (I, V) are negligible compared to the constant and linear terms. Likewise, for a sufficiently small change in the current, I, flowing into the terminal, the change in the terminal voltage, V, would be given approximately by

$$\delta V \approx R_{\mathrm{X}} \delta I.$$

2.1 Incremental DP Transfer Functions of a Parallel Connection

Suppose that we have two nonlinear devices, X and Y, with incremental terminal driving-point conductances of G_{X} and G_{Y}, connected in parallel, as shown in Fig. 4. We would like to

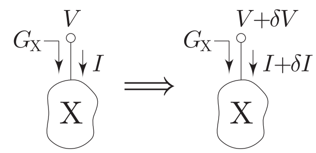

Figure 3: The incremental driving-point conductance looking into a terminal of a nonlinear device X is a measure of by how much the current flowing into the terminal changes, δI, in response to a small change in the voltage on the terminal, δV, and is given approximately by $\delta I/\delta V$. The incremental driving-point resistance looking into the same terminal is given approximately by $\delta V/\delta I$.

determine how the individual terminal driving-point conductances of X and Y combine in determining the equivalent incremental driving-point conductance of the parallel combination, G_{eq}. Now, Kirchhoff's current law (KCL) implies that

$$I = I_{\text{X}} + I_{\text{Y}}.$$

From this KCL equation, we can calculate the incremental driving-point conductance of the parallel combination of X and Y as

$$G_{\text{eq}} = \frac{\partial I}{\partial V} = \frac{\partial}{\partial V}\left(I_{\text{X}} + I_{\text{Y}}\right) = \frac{\partial I_{\text{X}}}{\partial V} + \frac{\partial I_{\text{Y}}}{\partial V} = G_{\text{X}} + G_{\text{Y}}.$$

Thus, the incremental driving-point conductance of the parallel combination of two nonlinear devices is given simply by the sum of the individual driving-point conductances. To obtain the incremental driving-point resistance of the parallel combination, we can take the reciprocal of the preceding equation to find that

$$R_{\text{eq}} = \frac{1}{G_{\text{eq}}} = \frac{1}{G_{\text{X}} + G_{\text{Y}}} = \frac{1}{(1/R_{\text{X}}) + (1/R_{\text{Y}})} = R_{\text{X}} \| R_{\text{Y}}.$$

Thus, incremental driving-point conductances and resistances of nonlinear devices connected in parallel combine like conductances and resistances ordinarily do.

We can also determine how the total current increment, δI, divides between X and Y in this situation. For a sufficiently small change in the node voltage, V, we have that

$$\delta I = G_{\text{X}}\delta V + G_{\text{Y}}\delta V$$

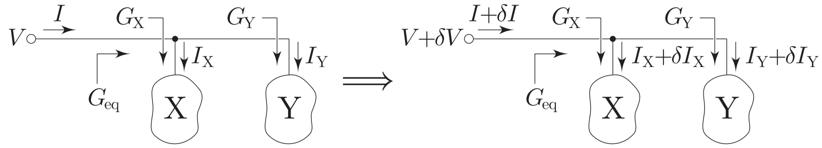

Figure 4: The incremental driving-point conductance of two nonlinear devices, X and Y, connected in parallel is given by the sum of the incremental driving-point conductances of the devices individually, $G_{\text{eq}} = G_{\text{X}} + G_{\text{Y}}$. The incremental driving-point resistance of such a configuration is given by $R_{\text{eq}} = R_{\text{X}} \| R_{\text{Y}}$.

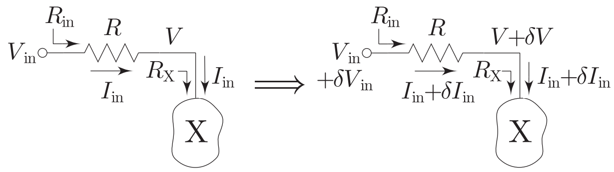

Figure 5: The equivalent incremental driving-point resistance of a resistor, R, connected in series with an incremental driving-point resistance, $R_{\rm X}$, is given by $R + R_{\rm X}$.

and that

$$\delta I_{\rm X} = G_{\rm X}\delta V \quad \text{and} \quad \delta I_{\rm Y} = G_{\rm Y}\delta V.$$

By solving the first equation for δV and substituting into each of the latter two equations, we have that

$$\delta I_{\rm X} = \frac{G_{\rm X}}{G_{\rm X} + G_{\rm Y}} \cdot \delta I \quad \text{and} \quad \delta I_{\rm Y} = \frac{G_{\rm Y}}{G_{\rm X} + G_{\rm Y}} \cdot \delta I.$$

Thus, the current increment divides itself between the devices in proportion to the incremental driving-point conductance of each, normalized by the total incremental driving-point conductance. Note that this is the same situation as a resistive current divider expressed in terms of conductances.

2.2 Incremental DP Transfer Functions of a Series Resistor

In this section, we shall determine the equivalent incremental driving-point resistance of a resistor, R, connected in series with a nonlinear device, X, with incremental driving-point resistance, $R_{\rm X}$, as illustrated in Fig. 5. Before we change the input voltage, $V_{\rm in}$, the situation is as shown on the left in Fig. 5. The input current, $I_{\rm in}$, is given by Ohm's law as

$$I_{\rm in} = \frac{V_{\rm in} - V}{R}.$$

This same current also flows into the terminal of X. Now, if we change $V_{\rm in}$ by $\delta V_{\rm in}$, the voltage across the resistor changes, which changes the current flowing in the resistor. Because this same current flows into X, the voltage V must also change in order for X to accommodate the new current. After the change in $V_{\rm in}$ and V, the current flowing through the resistor is given by Ohm's law as

$$I_{\rm in} + \delta I_{\rm in} = \frac{V_{\rm in} + \delta V_{\rm in} - (V + \delta V)}{R} = \underbrace{\frac{V_{\rm in} - V}{R}}_{I_{\rm in}} + \frac{\delta V_{\rm in} - \delta V}{R}.$$

By subtracting $I_{\rm in}$ from both sides of this equation, we find that the incremental input current, $\delta I_{\rm in}$, is given by

$$\delta I_{\rm in} = \frac{\delta V_{\rm in} - \delta V}{R},$$

which we can rearrange to find that

$$\delta V_{\rm in} = \delta I_{\rm in} R + \delta V. \tag{1}$$

For a sufficiently small change in V, the current increment δI_{in} will also be proportional to δV and given by

$$\delta I_{\text{in}} = \frac{\delta V}{R_{\text{X}}},$$

which we can rearrange to find that

$$\delta V = \delta I_{\text{in}} R_{\text{X}}. \tag{2}$$

By substituting Eq. 2 into Eq. 1 and solving for δI_{in}, we find that

$$\delta I_{\text{in}} = \frac{\delta V_{\text{in}}}{R + R_{\text{X}}}, \tag{3}$$

which implies that the equivalent incremental driving-point resistance, R_{in}, of a resistor, R, connected in series with an incremental driving-point resistance, R_{X}, is given by $R + R_{\text{X}}$.

We can also determine the change in V by substituting Eq. 3 back into Eq. 2. Doing so, we find that

$$\delta V = \frac{R_{\text{X}}}{R + R_{\text{X}}} \cdot \delta V_{\text{in}},$$

which is simply an incremental version of the familiar voltage divider rule for resistors.